A Minimal representation of MAPs

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Representation/parameter(s) of random variables

- Uniform(a, b): Lower and upper bounds
- Binomial(n, p): number of trials and P(success for a trial)
- Geometric(p): P(success for a trial)
- Poisson(λ): (Average) rate of occurrence
- Exponential(λ): 1/survival time
- N(μ, σ^2): First two centered-moments (Mean and variance)
- Hypergeometric(N, M, n)

$\Rightarrow$ All of above are minimal representations.

Laplace transform of random variables

(LT of the random variable and prob. density function)
Let X be a non-negative real-valued r.v. with pdf $f_X(x)$. Then, the LT of the r.v. X, and also the LT of the $f(x)$, is

$$\mathrm{E}(e^{-sX}) \equiv \tilde{f}(s) = \int_0^\infty e^{-st} f(t) dt$$

(Moments of X)

$$\mathrm{E}(X^n) \equiv \frac{d^n}{ds^n} \tilde{f}(s) \bigg|_{s=0}$$

- $X \sim$ Exponential(λ), $X \geq 0$

$$f(x) = \lambda e^{-\lambda x} \Rightarrow \tilde{f}(s) = \frac{\lambda}{\lambda + s} \Rightarrow \mathrm{E}(X^n) = n!/\lambda^n$$

Representation/parameter(s) of stochastic processes

- Poisson process(λ): i.i.d. exponential(λ) intervals
- Birth process($\lambda_0, \lambda_1, ...$): birth rates at each state
- Birth and death process($(\lambda_0, \mu_1), (\lambda_1, \mu_2), ...$): birth/death rates
- Renewal process: i.i.d. intervals

(Markov process/chain)

- Discrete-time Markov chain: 1-step transition matrix $\mathbf{P}^{n \times n}$
- Continuous-time Markov chain: infinitesimal generator $\mathbf{Q}^{n \times n}$
  - inter-event time in state $i \sim$ exponential(λ_i)
  - inter-event time and the next state are independent
- Semi-Markov process
  - generalization of CTMC with non-exponential sojourn times

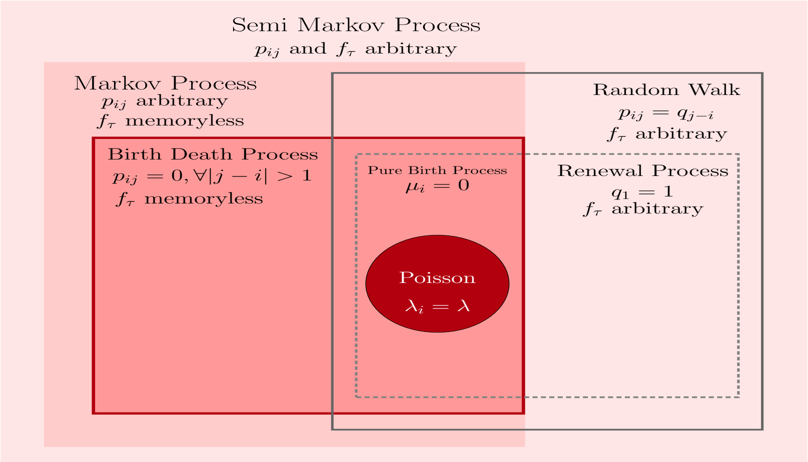


Figure: Categories of stochastic processes*

*Originally from Queueing Systems, Volume I: Theory by L. Kleinrock. Adapted by RK available at https://radhakrishna.typepad.com/queueing-systems.pdf

Overview

- MAP(n): Markovian arrival process of order n,
  - A composite of n Poisson processes
  - The minimal number of parameters: n^2
- Representations of MAPs
  - Markovian representation: $(\mathbf{D}_0, \mathbf{D}_1)$
  - Moments' representation: n^2 moments
  - Laplace transform (LT) representation
  - Jordan representation: $(\mathbf{E}, \mathbf{R})$
  - Minimal realization problem (MRP) representation: $(\mathbf{K}', \mathbf{R}')$
  - Characteristic polynomial representation
- LT of stationary intervals
  - A MAP(n) is fully described by a lag-1 joint LT
  - A rational function with $n^2 + n$ coefficients.
- Main result: Lag-1 joint LT can be written in n^2 parameters.

Markovian arrival processes

- How does an arrival takes place in MAP(n)s?
  - At any time point, the MAP(n) is in one of n phases.
  - Rate matrices ($\mathbf{D}_0$, $\mathbf{D}_1$) governs transitions between phases
  - $\mathbf{D}_1$ is the rate matrix generating arrivals (and transitions if any).
  - Off-diagonal entries of $\mathbf{D}_0$ are transition rates without arrivals.
- Markovian representation of a MAP(2) with 6 rate parameters.

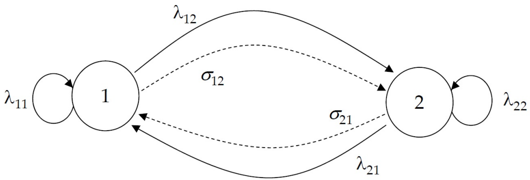
$$\mathbf{D}_0 = \begin{bmatrix} -\lambda_{11} - \lambda_{12} - \sigma_{12} & \sigma_{12} \\ \sigma_{21} & -\lambda_{21} - \lambda_{22} - \sigma_{21} \end{bmatrix}, \mathbf{D}_1 = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}$$

  $\mathbf{Q} \equiv \mathbf{D}_0 + \mathbf{D}_1$ is the infinitesimal generator for the CTMC.
- Special cases of MAPs
  - Poisson processes: exponential inter-arrival times
  - Erlang distribution and hyper-exponential distribution
  - Markov-modulated Poisson processes (MMPP)

Markovian arrival processes: MAP(2)

- Infinitesimal generator for the CTMC and transition diagram

$$\mathbf{Q} \equiv \mathbf{D}_0 + \mathbf{D}_1 = \begin{bmatrix} -\sigma_{12} - \lambda_{12} & \sigma_{12} + \lambda_{12} \\ \sigma_{21} + \lambda_{21} & -\sigma_{21} - \lambda_{21} \end{bmatrix},$$



  - Transition rate from phase 1 to 2: $\lambda_{12} + \sigma_{12}$
  - Transition rate from phase 2 to 1: $\lambda_{21} + \sigma_{21}$
  - Arrival rate in phase 1: $\lambda_{11} + \lambda_{12}$
  - Arrival rate in phase 2: $\lambda_{21} + \lambda_{22}$

Representations of MAP(n)s

- Markovian representation: $(\mathbf{D}_0, \mathbf{D}_1)$ matrices
  - A MAP(n) is described by two $n \times n$ transition rate matrices
  - $2n^2 - n$ parameters $\Rightarrow$ redundant (not minimal)!!!
  - Real-valued and straightforward
  - Not unique
- Moments' representation: n^2 moments
  - $2n - 1$ marginal moments
  - $(n-1)^2$ joint moments
  - Real-valued, minimal, and unique
  - Not straightforward $\Rightarrow$ Existence of a feasible $(\mathbf{D}_0, \mathbf{D}_1)$?
- LT representation: a rational function with $n^2 + n$ coefficients
  - A MAP(n) is fully described by a lag-1 joint LT
  - Real-valued and unique but not minimal
  - Not straightforward

(Question) Can we write lag-1 joint LT in terms of n^2 parameters?

Markovian representation $(\mathbf{D}_0, \mathbf{D}_1)$ of MAP(3)s

$$\mathbf{D}_0 = \begin{bmatrix} -\sigma_1 - \lambda_1 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & -\sigma_2 - \lambda_2 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & -\sigma_3 - \lambda_3 \end{bmatrix}, \ \mathbf{D}_1 = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}$$

with infinitesimal generator for the CTMC

$$\mathbf{Q} = \begin{bmatrix} -\sigma_1 - \bar{\lambda}_1 & \lambda_{12} + \sigma_{12} & \lambda_{13} + \sigma_{13} \\ \lambda_{21} + \sigma_{21} & -\sigma_2 - \bar{\lambda}_2 & \lambda_{23} + \sigma_{23} \\ \lambda_{31} + \sigma_{31} & \lambda_{32} + \sigma_{32} & -\sigma_3 - \bar{\lambda}_3 \end{bmatrix}$$

where $\sigma_i = \sum_{j \neq i} \sigma_{ij}, \lambda_i = \sum_{j=1}^{3} \lambda_{ij}, \bar{\lambda}_i = \sum_{j \neq i} \lambda_{ij}$.

- 6+9 = 15 rate parameters in $(\mathbf{D}_0, \mathbf{D}_1)$.
- Minimal number of parameters for MAP(3) is $3^2 = 9$.

Markovian representation $(\mathbf{D}_0, \mathbf{D}_1)$ of MAP(n)s

$$\mathbf{D}_0 = \begin{bmatrix} -\lambda_1 - \sigma_1 & \sigma_{1,2} & \cdots & \sigma_{1,n-1} & \sigma_{1,n} \\ \sigma_{2,1} & -\lambda_2 - \sigma_2 & \cdots & \sigma_{2,n-1} & \sigma_{2,n} \\ \vdots & & \ddots & & \vdots \\ \sigma_{n-1,1} & \sigma_{n-1,2} & \cdots & -\lambda_{n-1} - \sigma_{n-1} & \sigma_{n-1,n} \\ \sigma_{n,1} & \sigma_{n,2} & \cdots & \sigma_{n,n-1} & -\lambda_n - \sigma_n \end{bmatrix},$$

$$\mathbf{D}_1 = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1,n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n,1} & \lambda_{n,2} & \cdots & \lambda_{n,n} \end{bmatrix},$$

where $\lambda_i = \sum_{j=1}^{n} \lambda_{ij}$ and $\sigma_i = \sum_{j=1, j\neq i}^{n} \sigma_{ij}$.

- $2n^2 - n$ rate parameters in $(\mathbf{D}_0, \mathbf{D}_1)$.
- Minimal number of parameters for MAP(n) is n^2.

Characteristic polynomial equations

(Characteristic polynomial equation of $\mathbf{D}_0$ and $\mathbf{Q}$)

- $|s\mathbf{I} - \mathbf{D}_0| = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$
- $|s\mathbf{I} - \mathbf{Q}| = s^n + \Sigma_{n-1}s^{n-1} + \cdots + \Sigma_1 s$

(Coefficients of characteristic polynomial equations)

- $a_0 = (-1)^n|\mathbf{D}_0| = |-\mathbf{D}_0|$
- $a_{n-1} = \text{Trace}(-\mathbf{D}_0)$
- $|\mathbf{Q}| = 0 \Rightarrow \text{Constant term} = 0$
- $\Sigma_{n-1} = \text{Trace}(-\mathbf{Q})$
- $\Sigma_1 = \bar{q}_1 + \bar{q}_2 + \cdots + \bar{q}_n$ where $\bar{\boldsymbol{q}} = (\bar{q}_1, \bar{q}_2, ..., \bar{q}_n)$ is the vector of $(n-1) \times (n-1)$ principal minors of matrix $\mathbf{Q}$

Steady-state probability vectors of MAP(n)'s

(Stationary prob. vector $\boldsymbol{\pi}$ for CTMC with $\mathbf{Q}$)

- $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$ and $\boldsymbol{\pi}\boldsymbol{e} = 1$
- $\boldsymbol{\pi} = \bar{\boldsymbol{q}}/\Sigma_1$
- $\boldsymbol{\pi}\mathbf{Q} = \boldsymbol{\pi}(\mathbf{D}_0 + \mathbf{D}_1) = \mathbf{0} \Rightarrow \boldsymbol{\pi}(-\mathbf{D}_0) = \boldsymbol{\pi}\mathbf{D}_1$
- $(\mathbf{D}_0 + \mathbf{D}_1)\boldsymbol{e} = \mathbf{Q}\boldsymbol{e} = \mathbf{0} \Rightarrow -\mathbf{D}_0\boldsymbol{e} = \mathbf{D}_1\boldsymbol{e}$.
- Arrival rate: $\lambda_A \equiv \boldsymbol{\pi}\mathbf{D}_1\boldsymbol{e}$

(Stationary prob. vector $\boldsymbol{p}$ for embedded DTMC $\mathbf{P}$)

- $\mathbf{P} = -\mathbf{D}_0^{-1}\mathbf{D}_1$
- $\boldsymbol{p} = \boldsymbol{p}\mathbf{P}$ and $\boldsymbol{p}\boldsymbol{e} = 1$
- $\boldsymbol{p} = \boldsymbol{\pi}\mathbf{D}_1/\lambda_A$

Adjoint/adjugate/adjunct matrix

For an $n \times n$ matrix $\mathbf{A}$,

- The minor, M_{ij}, is the determinant of an $(n-1) \times (n-1)$ matrix obtained from $\mathbf{A}$ by deleting i-th row and j-th column.
- The cofactor: $C_{ij} = (-1)^{i+j} M_{ij}$.
- The cofactor matrix: $\mathbf{C} = [C_{ij}] = [(-1)^{i+j} M_{ij}]$
- The adjoint matrix: $\text{Adj}(\mathbf{A}) = \mathbf{C}^T$.
- $\mathbf{A}\text{Adj}(\mathbf{A}) = |\mathbf{A}|\mathbf{I} \qquad \Rightarrow \mathbf{A}^{-1} = \text{Adj}(\mathbf{A})/|\mathbf{A}|$

(Example)

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \mathbf{C} = \begin{pmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{pmatrix},$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, C_{12} = (-1)^{1+2} M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}.$$

Adjoint/adjugate/adjunct matrix

$$\mathrm{Adj}(\mathbf{A}) = \mathbf{C}^T = \begin{pmatrix} M_{11} & -M_{21} & M_{31} \\ -M_{12} & M_{22} & -M_{32} \\ M_{13} & -M_{23} & M_{33} \end{pmatrix}$$

$$= \begin{pmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}$$

- $\mathrm{Adj}(s\mathbf{I} - \mathbf{A}) = ?$

Cayley-Hamilton theorem

(**Cayley-Hamilton**) For an $n \times n$ matrix $\mathbf{D}$ with characteristic polynomial equation $|s\mathbf{I} - \mathbf{D}| = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$,

$$\mathbf{D}^n + a_{n-1}\mathbf{D}^{n-1} + \cdots + a_1\mathbf{D} + a_0\mathbf{I} = \mathbf{0}.$$

(Def) $\mathbf{C}_k \equiv \sum_{i=0}^{n-2-k} a_{k+i+1}\mathbf{D}_0^i + \mathbf{D}_0^{n-k-1}$ for $0 \le k \le n-2$

- $\mathbf{C}_0 = a_1\mathbf{I} + a_2\mathbf{D}_0 + \cdots + a_{n-1}\mathbf{D}_0^{n-2} + \mathbf{D}_0^{n-1}$
- $-\mathbf{D}_0\mathbf{C}_0 = -a_1\mathbf{D}_0 - a_2\mathbf{D}_0^2 - \cdots - a_{n-1}\mathbf{D}_0^{n-1} - \mathbf{D}_0^n = a_0\mathbf{I}$
- $\mathbf{C}_0 = \mathrm{Adj}(-\mathbf{D}_0)$ $\qquad \because -\mathbf{D}_0\mathrm{Adj}(-\mathbf{D}_0) = |-\mathbf{D}_0|\mathbf{I} = a_0\mathbf{I}$
- Set $\mathbf{C}_{n-1} \equiv \mathbf{I}$ and $\mathbf{C}_n \equiv \mathbf{0}$.

A minimal LT representation of MAP(n)s

(**Claim**) The lag-1 joint LT of MAP(n)s can be written in terms of n^2 parameters $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$

- $\boldsymbol{a} = (a_0, a_1, ..., a_{n-1})$: coefficients of $|s\mathbf{I} - \mathbf{D}_0|$
- $\boldsymbol{b} = (b_1, b_2, ..., b_{n-1})$: $b_k \equiv \boldsymbol{p}\mathbf{C}_k\mathbf{D}_1\boldsymbol{e}$ for $1 \le k \le n-1$.
- $\boldsymbol{c} = (c_{11}, ..., c_{n-1,n-1})$: $c_{ij} = \boldsymbol{p}\mathbf{C}_i\mathbf{D}_1\mathbf{C}_j\mathbf{D}_1\boldsymbol{e}$ for $i, j = 1, ..., n-1$
- $n + (n-1) + (n-1)^2 = n^2$ parameters

Since $\mathbf{C}_{n-1} \equiv \mathbf{I}$,

$$
\begin{aligned}
c_{i,n-1} &= \boldsymbol{p}\mathbf{C}_i\mathbf{D}_1^2\boldsymbol{e}, \\
c_{n-1,j} &= \boldsymbol{p}\mathbf{D}_1\mathbf{C}_j\mathbf{D}_1\boldsymbol{e}, \\
c_{n-1,n-1} &= \boldsymbol{p}\mathbf{D}_1^2\boldsymbol{e}.
\end{aligned}
$$

Constant term in the LT and $\mathrm{Adj}(s\mathbf{I} - \mathbf{D}_0)$

(**Lemma 1**) For MAP(n)s with $(\mathbf{D}_0, \mathbf{D}_1)$ and $\mathbf{C}_0 \equiv \mathrm{Adj}(-\mathbf{D}_0)$,

$$\begin{aligned} \boldsymbol{p}\mathbf{C}_0\mathbf{D}_1 &= |-\mathbf{D}_0|\boldsymbol{p}, \\ \mathbf{C}_0\mathbf{D}_1\boldsymbol{e} &= |-\mathbf{D}_0|\boldsymbol{e}. \end{aligned} \qquad \square$$

(**Lemma 2**)

$$\mathrm{Adj}(s\mathbf{I} - \mathbf{D}_0) \equiv \sum_{j=0}^{n-1} \sum_{i=0}^{n-j-1} a_{i+j+1} s^i \mathbf{D}_0^i = \sum_{i=0}^{n-1} s^i \mathbf{C}_i. \qquad \square$$

By Lemma 1,

$$\begin{aligned} \boldsymbol{p}\mathbf{C}_0\mathbf{D}_1\boldsymbol{e} &= |-\mathbf{D}_0|\boldsymbol{p}\boldsymbol{e} = a_0, \\ \boldsymbol{p}\mathbf{C}_0\mathbf{D}_1\mathbf{C}_0\mathbf{D}_1\boldsymbol{e} &= |-\mathbf{D}_0|^2\boldsymbol{p}\boldsymbol{e} = a_0^2. \end{aligned}$$

Constant term in the LT and Adj($s\mathbf{I} - \mathbf{D}_0$)

(Proof of Lemma 1) Since $\mathbf{P} \equiv (-\mathbf{D}_0)^{-1}\mathbf{D}_1$ and $\boldsymbol{p} = \boldsymbol{p}\mathbf{P}$,

$$\boldsymbol{p}\mathbf{C}_0\mathbf{D}_1 = \boldsymbol{p}\text{Adj}(-\mathbf{D}_0)\mathbf{D}_1 = \boldsymbol{p}|-\mathbf{D}_0|(-\mathbf{D}_0)^{-1}\mathbf{D}_1 = |-\mathbf{D}_0|\boldsymbol{p}.$$

Since $\mathbf{Q} \equiv \mathbf{D}_0 + \mathbf{D}_1$ and $\mathbf{Q}\boldsymbol{e} = \mathbf{0}$, we have $\mathbf{D}_1\boldsymbol{e} = -\mathbf{D}_0\boldsymbol{e}$ and

$$\mathbf{C}_0\mathbf{D}_1\boldsymbol{e} = \text{Adj}(-\mathbf{D}_0)(-\mathbf{D}_0)\boldsymbol{e} = |-\mathbf{D}_0|\boldsymbol{e}.$$

□

Marginal and joint LT of MAP(2) stationary intervals

$$\mathbf{D}_0 = \begin{bmatrix} -\lambda_1 - \sigma_1 & \sigma_1 \\ \sigma_2 & -\lambda_2 - \sigma_2 \end{bmatrix}, \mathbf{D}_1 = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix},$$
$$\mathbf{Q} = \begin{bmatrix} -\sigma_1 - \lambda_{12} & \sigma_1 + \lambda_{12} \\ \sigma_2 + \lambda_{21} & -\sigma_2 - \lambda_{21} \end{bmatrix},$$

where $\lambda_i = \lambda_{i1} + \lambda_{i2}$.

$$\begin{aligned} \tilde{f}(s) &= \boldsymbol{p}(s\mathbf{I} - \mathbf{D}_0)^{-1}\mathbf{D}_1\boldsymbol{e} \\ &= \frac{b_1 s + a_0}{s^2 + a_1 s + a_0}, \\ \tilde{f}(s,t) &= \boldsymbol{p}(s\mathbf{I} - \mathbf{D}_0)^{-1}\mathbf{D}_1(t\mathbf{I} - \mathbf{D}_0)^{-1}\mathbf{D}_1\boldsymbol{e} \\ &= \frac{c_{11}st + a_0 b_1(s+t) + a_0^2}{(s^2 + a_1 s + a_0)(t^2 + a_1 t + a_0)}. \end{aligned}$$

Marginal LT of MAP(n) inter-arrival times

(**Proposition 1**) The marginal LT of the stationary interval T of a MAP(n) is

$$\tilde{f}(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_2 s^2 + b_1 s + a_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0}.$$

Proof

$$\tilde{f}(s) \equiv \mathrm{E}(e^{-sT}) = \boldsymbol{p}(s\mathbf{I} - \mathbf{D}_0)^{-1}\mathbf{D}_1\boldsymbol{e}$$

$$= \frac{\boldsymbol{p}\mathrm{Adj}(s\mathbf{I} - \mathbf{D}_0)\mathbf{D}_1\boldsymbol{e}}{|s\mathbf{I} - \mathbf{D}_0|} = \frac{\boldsymbol{p}\sum_{k=0}^{n-1} s^k \mathbf{C}_k \mathbf{D}_1 \boldsymbol{e}}{|s\mathbf{I} - \mathbf{D}_0|}$$

since $\boldsymbol{p}\mathbf{C}_0\mathbf{D}_1\boldsymbol{e} = a_0$ and $\boldsymbol{p}\mathbf{C}_k\mathbf{D}_1\boldsymbol{e} = b_k$ for $1 \le k \le n-1$.

Lag-1 joint LT of MAP(n) interarrival times

(**Proposition 2**) The joint LT of two consecutive stationary intervals (T_1, T_2) of a MAP(n) is

$$\tilde{f}(s,t) = \frac{\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} c_{ij} s^i t^j + a_0 \sum_{i=1}^{n-1} b_i (s^i + t^i) + a_0^2}{\left(s^n + \sum_{i=0}^{n-1} a_i s^i\right)\left(t^n + \sum_{i=0}^{n-1} a_i t^i\right)}.$$

Proof

$$\begin{aligned}\tilde{f}(s,t) &\equiv \mathrm{E}(e^{-sT_1} e^{-tT_2}) = \boldsymbol{p}(s\mathbf{I} - \mathbf{D}_0)^{-1}\mathbf{D}_1(t\mathbf{I} - \mathbf{D}_0)^{-1}\mathbf{D}_1\boldsymbol{e} \\ &= \frac{\boldsymbol{p}\mathrm{Adj}(s\mathbf{I} - \mathbf{D}_0)\mathbf{D}_1\mathrm{Adj}(t\mathbf{I} - \mathbf{D}_0)\mathbf{D}_1\boldsymbol{e}}{|s\mathbf{I} - \mathbf{D}_0||t\mathbf{I} - \mathbf{D}_0|}\end{aligned}$$

Lag-1 joint LT of MAP(n) interarrival times

(Proof continued) The numerator can be written as

$$\boldsymbol{p}\left(\sum_{i=0}^{n-1} s^i \mathbf{C}_i\right) \mathbf{D}_1 \left(\sum_{i=0}^{n-1} t^i \mathbf{C}_i\right) \mathbf{D}_1 \boldsymbol{e}$$

$$= \boldsymbol{p}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \mathbf{C}_i\mathbf{D}_1\mathbf{C}_j\mathbf{D}_1\boldsymbol{e}s^it^j + \boldsymbol{p}\sum_{i=1}^{n-1}(\mathbf{C}_i\mathbf{D}_1\mathbf{C}_0\mathbf{D}_1 s^i + \mathbf{C}_0\mathbf{D}_1\mathbf{C}_i\mathbf{D}_1 t^i)\boldsymbol{e}$$

$$+ \boldsymbol{p}\mathbf{C}_0\mathbf{D}_1\mathbf{C}_0\mathbf{D}_1\boldsymbol{e}$$

$$= \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} c_{ij}s^it^j + a_0\sum_{i=1}^{n-1} b_i(s^i + t^i) + a_0^2$$

since $c_{ij} \equiv \boldsymbol{p}\mathbf{C}_i\mathbf{D}_1\mathbf{C}_j\mathbf{D}_1\boldsymbol{e}$ and $\boldsymbol{p}\mathbf{C}_i\mathbf{D}_1\mathbf{C}_0\mathbf{D}_1\boldsymbol{e} = \boldsymbol{p}\mathbf{C}_0\mathbf{D}_1\mathbf{C}_i\mathbf{D}_1\boldsymbol{e} = a_0b_i$.

Conclusion

- Markovian representation: $(\mathbf{D}_0, \mathbf{D}_1)$
  - Real-valued and straightforward
  - Redundant (not minimal) and not unique
- Moments' representation: n^2 moments
  - $2n - 1$ marginal moments and $(n-1)^2$ joint moments
  - Real-valued, minimal, and unique
  - Not straightforward $\Rightarrow$ Existence of a feasible $(\mathbf{D}_0, \mathbf{D}_1)$?
- LT representation: a rational function with $n^2 + n$ coefficients
  - A MAP(n) is fully described by a lag-1 joint LT
  - Real-valued and unique
  - Not straightforward

(Main result)

- $n^2 + n$ coefficients in the Lag-1 joint LT can be written in terms of n^2 parameters. $\Rightarrow$ A minimal representation!